Computation of Yukawa Couplings for Calabi-Yau Hypersurfaces in Weighted Projective Spaces

Yakov Kanter
F.R. Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853

Greene, Morrison and Plesser [1] have recently suggested a general method for constructing a mirror map between a d-dimensional Calabi-Yau hypersurface and its mirror partner for $d > 3$. We apply their method to smooth hypersurfaces in weighted projective spaces and compute the Chern numbers of holomorphic curves on these hypersurfaces. As anticipated, the results satisfy nontrivial integrality constraints. These examples differ from those studied previously in that standard methods of algebraic geometry which work in the ordinary projective space case for low degree curves are not generally applicable. In the limited special cases in which they do work we can get independent predictions, and we find agreement with our results.

I. INTRODUCTION

The mirror symmetry conjecture is increasingly being recognized as a powerful computational tool. One class of problems to which it has been successfully applied is the calculation of Chern numbers n_j of the parameter spaces of holomorphic degree j curves on certain special Calabi-Yau $d-$folds. Conformal field theory relates n_j's in a unique way to integer coefficients of certain $q-$expansions of Yukawa couplings on these Calabi-Yau manifolds. Mirror symmetry then allows one to calculate the Yukawa couplings by reinterpreting them in terms of deformations of the Hodge structure of the *mirror manifold*, thus effectively calculating n_j for all j. This was first done in [2] for the quintic Calabi-Yau 3-fold and its mirror. This technology is particularily easy to apply to smooth hypersurfaces in $\mathbf{P}^{d+1}$ or $\mathbf{WP}^{d+1}$ with one-dimensional Käler moduli spaces. (A more difficult case of multidimensional moduli spaces is discussed in [3] [4] [5] [6]). Traditional methods of algebraic geometry can calculate n_j's for $j \leq 3$ for projective space, giving exactly the numbers predicted by mirror symmetry. This makes it desirable to extend the list of Calabi-Yau manifolds for which Yukawa couplings have been explicitly calculated. In [1] this was accomplished for all smooth CY-hypersurfaces in $\mathbf{P}^{d+1}$ for $4 \leq d \leq 9$. In this paper we will compute Yukawa couplings for all hypersurfaces with Picard number equal to one in weighted projective spaces $\mathbf{WP}^{d+1}$ for $4 \leq d \leq 7$. It should be noted that while there is exactly one CY-hypersurface of this type in $\mathbf{P}^{d+1}$ for every $d \geq 3$, there are usually several such manifolds for weighted projective spaces. Furthermore, these examples are generally not accessible to traditional methods of algebraic geometry even for low degree curves and hence they further emphasize the power of physical methods.

II. SMOOTH HYPERSURFACES IN WP^{D+1}

In this section we review very briefly the mirror symmetry conjecture in the form proposed by the authors of [1] and describe their algorithm for computing Yukawa couplings as it applies to smooth hypersurfaces in weighted projective spaces. All of the material in this section is quite standard (see e.g. [1] and reviews in [20]). It is included here to make our presentation more self-contained.

A. Some background on mirror symmetry

Consider a superconformal Landau-Ginsburg model with superpotential $\mathcal{Q}$. Take $\mathcal{Q}$ to be a quasi-homogeneous polynomial. An argument due to Greene, Vafa and Warner [14], Martinec [15] and Witten [10] shows that this theory is an "analytic continuation" to negative Kähler class of a σ-model with Calabi-Yau manifold as a target space. This manifold can be described as a codimension one variety

$$\mathcal{W} = \{x, \mathcal{Q}(x) = 0\} \tag{1}$$

in a weighted projective space.

Truly marginal operators with $U(1)$ charge equal to $+1$ (-1) in the superconformal theory correspond to complex structure (Kähler) modes of the Calabi-Yau theory. We will be also considering "extended Kähler moduli spaces"

$H^p(X, \Lambda^p T^*)$ and "extended complex structure moduli spaces" $H^p(X, \Lambda^p T)$. Following Witten [9], the couplings involving elements of $H^p(X, \Lambda^p T^*)$ will be called **A**-model couplings, while couplings involving elements of $H^p(X, \Lambda^p T)$ – **B**-model couplings.

Now consider the action of the maximal group of scaling symmetries G on both Landau-Ginsburg and Calabi-Yau models for $\mathcal{Q}$ of Fermat type. It is well known (see, e.g. review [12] and references therein) that orbifolding the LG theory by G simply simply changes the sign of one of the $U(1)$ charges and gives a theory isomorphic to the original one. For a CY model, dividing by G means taking the orbifold of the target space. The unique connection with a LG model now leads to the central result of the mirror symmetry, namely, to the prediction that the two resulting quantum σ-models are isomorphic (even though they have different tree-level actions, since their target manifolds will in general be topologically distinct). In particular, all n-point functions computed in both σ-models must be equal for corresponding modes.

This result can be used to calculate Chern numbers of holomorphic curves on a variety $\mathcal{W}$ as first shown in [2]. Consider a Yukawa coupling between elements of $H^p(\mathcal{W}, \Lambda^p T^*)$. It was shown in [16] [17] [18] [2] [19] that for the case of a 1-dimensional Kähler moduli space

$$\langle O^i O^j O^k \rangle = \langle e_i\, e_j\, e_k \rangle + \sum_l \frac{q^l}{1-q^l} \cdot n_l, \tag{2}$$

where O^j are operators corresponding to elements of $H^p(\mathcal{W}, \Lambda^p T^*)$, $q = e^{2\,\pi\, i\, t}$, t is a $\mathbb{Z}$-perodic local coordinate on the complexified Kähler cone of $\mathcal{W}$ and n_l's are the Chern numbers of degree l curves on $\mathcal{W}$. Mirror symmetry applied to 3-point functions allows one to interpret equation (2) as a coupling between elements of $H^p(\mathcal{M}, \Lambda^p T)$, where $\mathcal{M}$ is the mirror partner of $\mathcal{W}$. In this interpretation $\langle O^i O^j O^k \rangle$ is given by the cup product of elements of $H^p(\mathcal{M}, \Lambda^p T)$ and is not corrected by instantons [13]. Thus it can be computed using methods of algebraic geometry as a function of z, where z is some local coordinate on the moduli space of complex structures of $\mathcal{M}$. The link to equation (2) will be established once we know t as a function of z. More generally, mirror symmetry predicts that there exists a one-to-one map between $H^p(\mathcal{M}, \Lambda^p T)$ and $H^p(\mathcal{W}, \Lambda^p T^*)$ which we will call the *generalized mirror map*, since unlike the 3-dimensional case it is *not* determined by $z \mapsto t(z)$. The preceding discussion shows that our problem of calculating the Chern numbers of the holomorphic curves on $\mathcal{W}$ can be reduced to:

- Computing properly normalized Yukawa couplings on the mirror partner of $\mathcal{W}$
- Computing the generalized mirror map.

The authors of [1] suggested a solution [1] to both problems by conjecturing that the generalized mirror map takes the primary vertical subspace of $\bigoplus_p H^{p,p}(\mathcal{W}, \mathbb{Z})$ into the horizontal subspace of the Gauss-Manin connection on $\bigoplus_p H^p(\mathcal{W}, \Lambda^p T)$. Specifically, they showed that one can construct the mirror map in the following way :

1. Select a primary vertical subbasis $e_0, e_1, \cdots, e_d$ of $\bigoplus_p H^{p,p}(\mathcal{W}, \mathbb{Z})$, satisfying the operator product equations

$$e_1 \cdot e_{j-1} = c^{-1}\, A_{j-1}(t)\, e_j, \tag{3a}$$

$$\langle e_i, e_j \rangle = \eta^{(i,j)} = c\, \delta_{i+j,d}, \tag{3b}$$

$$e_1 \cdot e_d = 0. \tag{3c}$$

2. Declare that the mirror of this basis is a set of elements $\alpha_0,\ \alpha_1, \cdots, \alpha_d$ of $\bigoplus_p H^{p,d-p}(\mathcal{M}, \mathbb{C})$ with $\alpha_0 = \Omega \in H^{p,0}(\mathcal{M}, \mathbb{C})$ such that the same axioms hold for α_j with **B**-model three-point function $B_{j-1}(z)$ replacing $A_{j-1}(t)$ in equation (3a) and action of Gauss-Manin connection replacing operator product

$$e_1 \cdot e_{j-1} = c^{-1}\, B_{j-1}(z)\, e_j, \tag{4a}$$

$$\langle e_i, e_j \rangle = \eta^{(i,j)} = c\, \delta_{i+j,d}, \tag{4b}$$

---

[1] Apparently their results can be justified using the methods of [24]

$$e_1 \cdot e_d = 0. \tag{4c}$$

They further showed that conditions of (4) will be satisfied for α_j's which are covariantly constant with respect to the Gauss-Manin connection and such that their period matrix

$$P_{\mu i} = \int_{\gamma_\mu} \alpha_i \tag{5}$$

is upper-triangular with ones on the diagonal for homology cycles $\gamma_0,\, \gamma_1, \cdots, \gamma_d$ satisfying the maximum unipotency condition at $z = 0$ (see equation (9c) for a definition of z). This allows one to compute the mirror map and normalized Yukawa couplings using the algorithm described in the next subsection.

B. An Algorithm for Computing the Mirror Map

Let $\mathcal{W}_t$ be a Fermat hypersurface in $\mathbf{WP}(k_1, \ldots, k_{d+1})$ defined by the vanishing loci of

$$Q(x) = x_1^{d_1} + \cdots + x_{d+1}^{d_{d+1}} \tag{6}$$

with complex structure induced from WP^d and a $t-$dependent Kähler structure induced from

$$e(t) = t \cdot e, \tag{7}$$

where e is some fixed generator of $H^{1,1}(\mathbf{WP}^d, \mathbb{Z})$. Now consider

$$\tilde{\mathcal{M}_\psi} = \{x \in \mathbf{WP}(k_1, \ldots, k_{d_{d+1}}) \,|\, Q(x, \psi) = 0\}, \tag{8}$$

where

$$Q(x, \psi) = x_1^{d_1} + \cdots + x_{d+1}^{d_{d+1}} - k\psi \cdot x_1 \cdots x_{d+1}, \tag{9a}$$

$$k = \sum_j k_j, \tag{9b}$$

$$z = \psi^{-k}. \tag{9c}$$

If we choose the weights k_j and degrees d_j so that $gcd(k_i, k_j)$=1 for $i \neq j$, $d_j = k/k_j$,
then $\mathbf{WP}(k_1, \ldots, k_{d+1})$ will be smooth away from the origin and by Lefshetz's theorem $h^{1,1}(\tilde{\mathcal{M}_\psi}) = 1$ when $\tilde{\mathcal{M}_\psi}$ is non-singular.

Let $\mathcal{M}_\psi = \tilde{\mathcal{M}_\psi}/G$, where G is the maximal group of scaling symmetries [12], with $\psi-$dependent complex structure induced from $\mathbf{WP}^d$ and the Kähler structure corresponding to the Landau-Ginsburg point in the Kähler moduli space [10] [11] . Mirror symmetry now predicts that for every $\psi \in \mathbb{C}$ there exists $t(\psi) \in \mathbb{C}$ such that $\mathcal{M}_\psi$ is a mirror of $\mathcal{W}_{t(\psi)}$, i.e. that quantum $\sigma-$models with $\mathcal{M}_\psi$ and $\mathcal{W}_{t(\psi)}$ as target manifolds are isomorphic. The map $\psi \mapsto t(\psi)$ together with Yukawa couplings can be obtained as a part of the generalized mirror map in the following way.

Step 1.
A form $\alpha \in H^{p,d-p}(\mathcal{M}_\psi, \mathbb{C})$ is covariantly constant with respect to Gauss-Manin connection if

$$f(z) = \int_{\gamma(z)} \alpha \tag{10}$$

satisfies a d-th order ordinary differential equation, called the Picard-Fuchs equation [23]. An equivalent system of first-order equations is

$$z\, \frac{d\, w}{d\, z} = A(z)\, w(z), \tag{11}$$

where

$$w(z) = \begin{pmatrix} \int_\gamma \alpha \\ z\,\frac{d}{d\,z}\int_\gamma \alpha \\ \vdots \\ (z\,\frac{d}{d\,z})^d \int_\gamma \alpha \end{pmatrix}, \tag{12}$$

$$A(z) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ B_0(z) & \cdots & & B_d(z) \end{pmatrix} \tag{13}$$

The coefficients $B_0, \ldots, B_d$ encode all the information necessary to compute Yukawa couplings for a specific hypersurface. They can be generated using Griffiths' pole reduction method, which was described in [7] for $d = 3$ but works for $d \geq 3$ without modification. Just as in the $d = 3$ case the problem reduces to finding an explicit representation of polynomials in a Jacobian ideal over $\mathbb{Q}(\psi)$. This is achieved in the following way.
Suppose we want to find A_j's such that

$$g = \sum_j A_j \cdot F_j, \tag{14}$$

where g and F_j's are known polynomials. First, we use the built-in ability of MACAULAY [21] to compute syzygies to find A_j over a finite field $\mathbb{F}_{31991} = \mathbb{Z}/31991 \cdot \mathbb{Z}$. In general, if $\mathcal{I}$ is an ideal with generators $f_1, \ldots, f_n$, then a syzygy of $\mathcal{I}$ is a n-dimensional vector $(g_1, \cdots, g_n)$, such that $\sum_{i=1}^n g_i \cdot f_i = 0$. Thus to find a representation (14) we simply find a set of generators for the syzygy module of $\mathcal{J} = \langle g, F_1, \ldots, F_n \rangle$ and pick the one with a constant first element. Then all monomial coefficients of A_j are replaced with unknown constants and the result substituted back into equation (14). The resulting linear equations for these constants are then solved in MAPLE over $\mathbb{Q}(\psi)$, thus lifting the representation to $\mathbb{Q}(\psi)$.

In all the examples we considered, B_j's turned out to be rational functions of z, as expected, and

$$B_0(0) = \ldots = B_d(0) = 0 \tag{15}$$

The last condition ensures that $z = 0$ is a regular singular point of (11) and that the monodromy is maximally unipotent at $z = 0$.

Step 2.
A fundamental matrix of solutions of (11) can be written in the form

$$\begin{aligned} \Phi(z) &= S(z)\, z^{A(0)} \\ &= S(z) \begin{pmatrix} 1 & \log(z) & & \frac{1}{n!}(\log(z))^n \\ & \ddots & \ddots & \\ & & & \log(z) \\ 0 & & & 1 \end{pmatrix} \end{aligned} \tag{16}$$

where $S(z)$ is a single-valued holomorphic matrix-valued function of z. Substitution of (16) into (11) gives an equation for S,

$$z\,\frac{dS}{dz} + S(z) \cdot A(0) = A(z) \cdot S(z), \tag{17}$$

which can be solved using power series techniques.

Step 3.
Put the matrix S in the upper triangular form using only row operations. Call the resulting $(d+1) \times (d+1)$ matrix T and let the indices of T run from 0 to d.

The canonical parameter t of [8] is then given by

$$t = T_{01} \tag{18}$$

and the canonical variable of the $q-$expansion is, as usual,

$$q = \exp(2\pi i\, t) \tag{19}$$

Step 4.
Fundamental Yukawa couplings, i.e. Yukawa couplings involving at least one element of $H^{-1,1}(\mathcal{M}_\psi)$ when interpreted as couplings in **B**-model of [9], can be expressed in terms of the first superdiagonal of T. Following [1] we shall denote $< O^1\, O^j\, O^{d-j-1} >$ as Y^1_j . Then it was shown in [1] that all couplings can be expressed in terms of the fundamental ones using the associativity of the operator product expansion and that

$$Y^1_j = c\,\frac{1 + z\,\partial_z T_{j,j+1}}{1 + z\,\partial_z T_{0,1}}, \tag{20}$$

where c is the degree of the surface, i.e. the smallest d_j in equation (9a).

III. EXAMPLES

The authors of [1] compute Yukawa couplings for all smooth Picard one Calabi-Yau hypersurfaces in $\mathbf{P}^{d+1}$ for $4 \le d \le 10$. There is exactly one such hypersurface for each dimension d. In this section we will compute the fundamental Yukawa couplings for all Picard one hypersurfaces in $\mathbf{WP}^{d+1}$ for $4 \le d \le 7$.

A. Hypersurfaces in WP^5

There is exactly one set of $(k_1, \ldots, k_6)$ that satisfies the necessary conditions for the smoothness of $\mathbf{WP}(k_1, \ldots, k_6)$, namely (5,1,1,1,1,1). Thus the only family of Calabi-Yau submanifolds is given by

$$Q = x_1^2 + x_2^{10} + \cdots + x_6^{10} - 10\,\psi\, x_1 \cdots x_6 = 0 \tag{21}$$

There is only one independent 3-point function Y^1_1 in this case. We find

$$Y^1_1 = 2 + 1582400\frac{q}{1-q} + 3167779945600\frac{q^2}{1-q^2} + 7052557179599697600\frac{q^3}{1-q^3} + \ldots \tag{22}$$

B. Hypersurfaces in $WP^6 - WP^8$

All smooth hypersurfaces in WP^{d+1} $5 \le d \le 7$ are described in tables I – III. The fundamental Yukawa couplings are summarized in tables IV – V. Methods of algebraic geometry allow one to compute Yukawa couplings for weighted projective spaces with all but one weight equal to 1. In particular, Sheldon Katz has confirmed our result for Y^2_1 in $\mathbf{WP}^6(2,1,1,1,1,1)$ [22]

IV. CONCLUSIONS

In this paper we used the method described in [1] to calculate normalized Yukawa couplings for Calabi-Yau hypersurfaces of complex dimensions $d = 4, \ldots, 7$ in weighted projective spaces. The fact that the coefficients of Yukawa couplings' expansions in canonical variable q turn out to be integers satisfying rather intricate divisibility constraints described in [1] provides support for the mirror symmetry conjecture in this case. We have found agreement between our results and those of more standard mathematical techniques when the latter are applicable. Finally, the general case of a Calabi-Yau hypersurface in weighted projective space is not accessible to standard methods of algebraic geometry and hence extends the domain of cases which can only be analyzed with physical methods.

V. ACKNOWLEDGMENTS

It is a pleasure to acknowledge very helpful discussions with Bill Dimm, Brian Greene, Mike Stillman and Bernd Strumfels. This work was partially supported by a grant from the National Science Foundation.

TABLE I. Smooth hypersurfaces of 6-dimensional weighted projective spaces

$(k_1, \cdots k_7)$	$k = \sum_{j=1}^{7} k_j$	Defining equation in WP^6
(2,1,1,1,1,1,1)	8	$x_1^4 + x_2^8 + \cdots + x_7^8 - 8\,\psi\, x_1 \cdots x_7 = 0$
(3,1,1,1,1,1,1)	9	$x_1^3 + x_2^9 + \cdots + x_7^9 - 9\,\psi\, x_1 \cdots x_7 = 0$
(6,1,1,1,1,1,1)	12	$x_1^2 + x_2^{12} + \cdots + x_7^{12} - 12\,\psi\, x_1 \cdots x_7 = 0$
(4,3,1,1,1,1,1)	12	$x_1^3 + x_2^4 + x_3^{12} + \cdots + x_7^{12} - 12\,\psi\, x_1 \cdots x_7 = 0$
(7,2,1,1,1,1,1)	14	$x_1^2 + x_2^7 + x_3^{14} + \cdots + x_7^{14} - 14\,\psi\, x_1 \cdots x_7 = 0$

TABLE II. Smooth hypersurfaces of 7-dimensional weighted projective spaces

$(k_1, \cdots k_8)$	$k = \sum_{j=1}^{8} k_j$	Defining equation in WP^7
(7,1,1,1,1,1,1,1)	14	$x_1^2 + x_2^{14} + x_3^{14} + \cdots + x_8^{14} - 14\,\psi\, x_1 \cdots x_8 = 0$

TABLE III. Smooth hypersurfaces of 8-dimensional weighted projective spaces

$(k_1, \cdots k_9)$	$k = \sum_{j=1}^{9} k_j$	Defining equation in WP^8
(2,1,1,1,1,1,1,1,1)	10	$x_1^5 + x_2^{10} + \cdots + x_9^{10} - 10\,\psi\, x_1 \cdots x_9 = 0$
(4,1,1,1,1,1,1,1,1)	12	$x_1^3 + x_2^{12} + \cdots + x_9^{12} - 12\,\psi\, x_1 \cdots x_9 = 0$
(5,3,1,1,1,1,1,1,1)	15	$x_1^3 + x_2^5 + x_3^{15} + \cdots + x_9^{15} - 15\,\psi\, x_1 \cdots x_9 = 0$
(8,1,1,1,1,1,1,1,1)	16	$x_1^2 + x_2^{16} + x_3^{16} + \cdots + x_9^{16} - 16\,\psi\, x_1 \cdots x_9 = 0$
(9,2,1,1,1,1,1,1,1)	18	$x_1^2 + x_2^9 + x_3^{18} + \cdots + x_9^{18} - 18\,\psi\, x_1 \cdots x_9 = 0$

TABLE IV. Fundamental 3-point functions for smooth hypersurfaces in WP^6

(2,1,1,1,1,1,1)	$Y_1^1 = 4 + 3049216\,\frac{q}{1-q} + 7472581386752\,\frac{q^2}{1-q^2} + 21454661245363681536\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 4 + 5379584\,\frac{q}{1-q} + 16429968863232\,\frac{q^2}{1-q^2} + 55188836029204818432\,\frac{q^3}{1-q^3} + \ldots$
(3,1,1,1,1,1,1)	$Y_1^1 = 3 + 8022402\,\frac{q}{1-q} + 66933014780124\,\frac{q^2}{1-q^2} + 657680002962846783606\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 3 + 13973877\,\frac{q}{1-q} + 146713683733290\,\frac{q^2}{1-q^2} + 1692469099684864365660\,\frac{q^3}{1-q^3} + \ldots$
(6,1,1,1,1,1,1)	$Y_1^1 = 2 + 71754624\,\frac{q}{1-q} + 7975093545660672\,\frac{q^2}{1-q^2} +$ $1044039441585542582459520\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 2 + 126008064\,\frac{q}{1-q} + 17658771986147328\,\frac{q^2}{1-q^2} +$ $2719158056221778746705152\,\frac{q^3}{1-q^3} + \ldots$
(4,3,1,1,1,1,1)	$Y_1^1 = 3 + 765516096\,\frac{q}{1-q} + 568850164748055936\,\frac{q^2}{1-q^2} +$ $504680055676890191453456064\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 3 + 1400499072\,\frac{q}{1-q} + 1338726917072077056\,\frac{q^2}{1-q^2} +$ $1411413775244683001901233280\,\frac{q^3}{1-q^3} + \ldots$
(7,2,1,1,1,1,1)	$Y_1^1 = 2 + 1301207936\,\frac{q}{1-q} + 2541843733963905280\,\frac{q^2}{1-q^2} +$ $5895640558847778162251490432\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 2 + 2434050304\,\frac{q}{1-q} + 6058326351439047168\,\frac{q^2}{1-q^2} +$ $16648147547178442866316220160\,\frac{q^3}{1-q^3} + \ldots$

TABLE V. Fundamental 3-point functions for smooth hypersurfaces in WP^7

(7,1,1,1,1,1,1,1)		$Y_1^1 = 2 + 3237982720\,\frac{q}{1-q} + 21183078150223087616\,\frac{q^2}{1-q^2} +$ $175301491479186058292989251072\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 2 + 8106083328\,\frac{q}{1-q} + 78167476562234465280\,\frac{q^2}{1-q^2} +$ $846729903166068966083368713216\,\frac{q^3}{1-q^3} + \ldots$

TABLE VI. Fundamental 3-point functions for smooth hypersurfaces in WP^8

2,1,1, … 1		$Y_1^1 = 5 + 873342000\,\frac{q}{1-q} + 850904087051808000\,\frac{q^2}{1-q^2} +$ $1126679621495881973966658000\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 5 + 2788544000\,\frac{q}{1-q} + 4391772046773856000\,\frac{q^2}{1-q^2} +$ $8088656001785886785691456000\,\frac{q^3}{1-q^3} + \ldots$ $Y_3^1 := 5 + 3996616000\,\frac{q}{1-q} + 7539434669357184000\,\frac{q^2}{1-q^2} +$ $15863782567300449640323384000\,\frac{q^3}{1-q^3} + \ldots$
4,1,1, … 1		$Y_1^1 = 3 + 7258320576\,\frac{q}{1-q} + 96985196693399291904\,\frac{q^2}{1-q^2} +$ $1777393929237545056819269167424\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 3 + 23593099392\,\frac{q}{1-q} + 516294676633310908416\,\frac{q^2}{1-q^2} +$ $13238896229337709463174524669824\,\frac{q^3}{1-q^3} + \ldots$ $Y_3^1 = 3 + 33901042752\,\frac{q}{1-q} + 892907832989728097280\,\frac{q^2}{1-q^2} +$ $26228563889472641235997337881536\,\frac{q^3}{1-q^3} + \ldots$
8,1,1, … 1		$Y_1^1 = 2 + 152056940544\,\frac{q}{1-q} + 63766419662687527673856\,\frac{q^2}{1-q^2} +$ $36785354090785907492391715484153856\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 2 + 498854371328\,\frac{q}{1-q} + 344276495484191699451904\,\frac{q^2}{1-q^2} +$ $278600071980095768674546472262721536\,\frac{q^3}{1-q^3} + \ldots$ $Y_3^1 = 2 + 718234344448\,\frac{q}{1-q} + 597762641310376010833920\,\frac{q^2}{1-q^2} +$ $554788665268211922034998620987433984\,\frac{q^3}{1-q^3} + \ldots$
5,3,1, … 1		$Y_1^1 = 3 + 1004767067700\,\frac{q}{1-q} + 1935016728809768017084800\,\frac{q^2}{1-q^2}$ $+5206880462500493951270650091011932300\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 3 + 3609099738900\,\frac{q}{1-q} + 11658783428685286478013600\,\frac{q^2}{1-q^2}$ $+44401976701844742556180178323097636100\,\frac{q^3}{1-q^3} + \ldots$ $Y_3^1 = 3 + 5344739516475\,\frac{q}{1-q} + 21077586779051709641435400\,\frac{q^2}{1-q^2}$ $+92729708830552378723454519048709905400\,\frac{q^3}{1-q^3} + \ldots$
9,2,1, … 1		$Y_1^1 = 2 + 3306710034432\,\frac{q}{1-q} + 31348838504534376710504448\,\frac{q^2}{1-q^2}$ $+413657258502381181112268855777401837568\,\frac{q^3}{1-q^3} + \ldots$ $Y_2^1 = 2 + 11682851512320\,\frac{q}{1-q} + 185001959444922156143738880\,\frac{q^2}{1-q^2}$ $+3448977159195424871434590786027577221120\,\frac{q^3}{1-q^3} + \ldots$ $Y_3^1 = 2 + 17375270943744\,\frac{q}{1-q} + 334695839710556882880626688\,\frac{q^2}{1-q^2}$ $+7196484971475862667134042111055293694976\,\frac{q^3}{1-q^3} + \ldots$

[1] B.R. Greene, D.R. Morrison, M.R. Plesser: *Mirror Manifolds in Higher Dimension* CLNS-93/1253, IASSNS-HEP-94-2, YCTP-92.
[2] P. Candelas, X.C. de la Ossa, P. Green, L. Parkes Nucl. Phys. **B359**(1991), p.21
[3] P. Candelas, A. Font, S. Katz, D.R. Morrison: *Mirror Symmetry for Two Parameter Models - I* Nucl. Phys. **B416** (1994), p.481
[4] P. Candelas, A. Font, S. Katz, D.R. Morrison: *Mirror Symmetry for Two Parameter Models - II* University of Texas preprint UTTG-25-93.
[5] S. Hosono, A. Klemm, S. Theisen, S.T. Yau: *Mirror Symmetry, Mirror Map and Applications to Calabi-Yau Hypersurfaces*, Harvard University preprint HUMTP-93-0801
[6] S. Hosono, A. Klemm, S. Theisen, S.T. Yau: *Mirror Symmetry, Mirror Map and Applications to Complete intersections of Calabi-Yau Spaces*, Harvard University preprint HUMTP-94/02
[7] D.R. Morrison, in *Essays on Mirror Manifolds* (S.T.Yau, editor), International Press, 1992, p.241.
[8] D.R. Morrison, J.Amer.Math.Soc. **6** (1993) 223
[9] E. Witten, in *Essays on Mirror Manifolds* (S.T.Yau, editor), International Press, 1992, p.120.
[10] E. Witten: *Phases of N=2 theories in two dimensions*, IASSNS-HEP-93/3
[11] P.S. Aspinwall, B.R. Greene, D.R. Morrison: *Calabi-Yau Moduli Space, Mirror Manifolds and Spacetime Topology Change in String Theory*, IASSNS-HEP-93/38, CNLS-93/1236.
[12] B.R. Greene, M.R. Plesser: *An Introduction to Mirror Manifolds*, in *Essays on Mirror Manifolds* (S.T.Yau, editor), International Press, 1992, p.1
[13] J. Distler, B.R. Greene, Nucl.Phys. **B 309** (1988), p.295
[14] B.R. Greene, C. Vafa, N. Warner, Nucl.Phys. **B 324** (1989), p.371
[15] E. Martinec, in *V.G. Knizhnic Memorial Volume* (L.Brink et al.,eds)1989; Phys.Lett **B217**(1989) 431.
[16] A. Strominger, E. Witten, Commun. Math. Phys. **101**(1985),p.341
[17] A. Strominger, Phys.Rev. Lett. **55**(1985),p.2547
[18] M. Dine, N. Seiberg, X.-G. Wen, E. Witten, Nucl.Phys. **B278**(1987)p.769; Nucl.Phys. **B289**(1987)p.319
[19] P. Aspinwall, D.B. Morrison: *Topological Field Theory and Rational Curves*, Oxford preprint, to appear in Commun. Math. Phys.
[20] *Essays on Mirror Manifolds* (S.T.Yau, editor), International Press, 1992
[21] D. Bayer and M. Stillman, *Macaulay: A system for computation in algebraic geometry and commutative algebra*. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from *zariski.harvard.edu* via anonymous ftp.
[22] Sheldon Katz. Private communication.
[23] V.I. Arnold, A.N. Varchenko, S.M. Gusein-Zade, *Singularities of differentiable maps.* vol.2 §10. Birkhäuser, 1985
[24] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa: *Holomorphic Anomaly in Topological Field Theory*, Harvard preprint HUTP-93-A008.